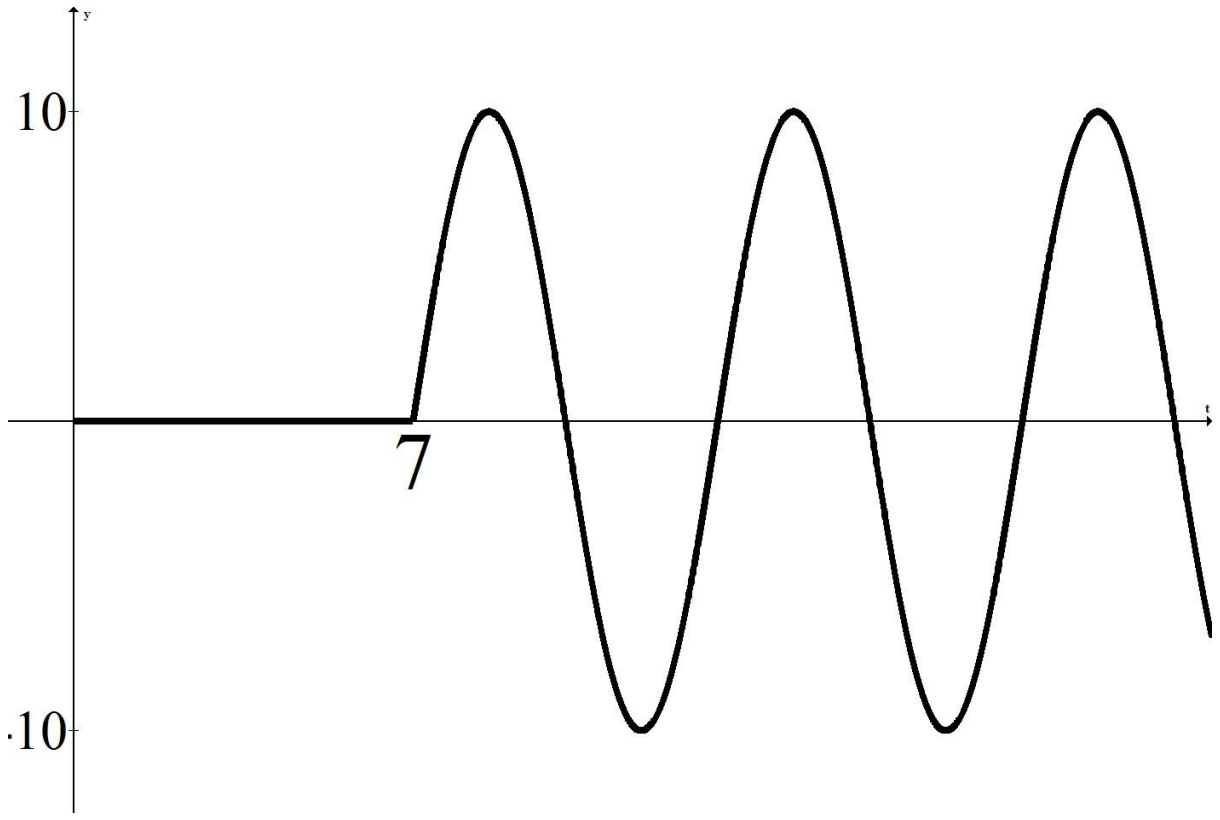
A Full Example of Discontinuous Forcing

Solve

$$y'' + 2y' + y = \begin{cases} 0 & ,0 \le t < 7; \\ 10\sin(t-7) & ,t \ge 7. \end{cases}$$

$$\text{with } y(0) = 0, y'(0) = 10$$

Here is what the forcing function looks like



1. Rewrite forcing in terms of step functions:

$$f(t) = 10u_7(t)\sin(t-7)$$

2. Laplace transform both sides:

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 10\mathcal{L}\{u_7(t)\sin(t-7)\}$$

3. Using Laplace rules, simplifying, and partial fractions we get

$$(s+1)^2\mathcal{L}\{y\} - 10 = \frac{10e^{-7s}}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{10e^{-7s}}{(s^2+1)(s+1)^2} + \frac{10}{(s+1)^2}$$

$$= 5e^{-7s}\left(\frac{-s}{s^2+1} + \frac{1}{s+1} + \frac{1}{(s+1)^2}\right) + \frac{10}{(s+1)^2}$$

4. The inverse Laplace transform gives

$$5u_7(t)\left(-\cos(t-7) + e^{-(t-7)} + (t-7)e^{-(t-7)}\right) + 10te^{-t}$$

Here is a graph of the solution: